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Note

A note on clone sets in representable matroids

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Abstract

We investigate the size of clone sets in representable matroids.

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1. Introduction

Recently clones have become important in the study of matroid representability [3–6]. We show that a sufficiently connected matroid that is representable over a small field does not have a large clone set.

The terminology follows [3,4,7]. Elements e and f in a matroid M are clones if interchanging e and f and fixing all other elements is an automorphism of M. A *clonal class* of M is a maximal set $X \subseteq E(M)$ such that each pair of elements of X are clones. Clonal classes of M include its set of loops, its set of coloops, each parallel class, and each series class. Such clonal classes are called *trivial clonal classes*. A *clone set* of M is a subset of a nontrivial clonal class that contains at least two elements. The clone sets of M and M^* coincide. If there is no single-element extension of M by e', so that $\{e, e'\}$ is a clone set, then e is *fixed* in M. We say that e is *cofixed* in M if it is fixed in M^*. Clone sets contain neither fixed nor cofixed elements.

The starting point for our research is the following result of Geelen et al. [3, Lemma 5.6].

Theorem 1.1. *A* 2*-connected binary matroid has no clone sets*.

We generalize Theorem 1.1 and give several consequences, the first of which was conjectured by Whittle (private communication). We conjecture that Theorem 1.2 holds for all q.

Theorem 1.2. *For* $q \leqslant 5$, *if* M *is a* 3*-connected* GF(q)*-representable matroid that contains a clone set* X *with at least* $q-1$ *elements*, *then* M *is uniform*.

Corollary 1.3. *If* M *is a* 3*-connected* GF(3)*-representable matroid that contains a clone set*, *then* $M \cong U_{2,4}$.

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Corollary 1.4. *If M is a* 3-*connected* GF(4)-*representable matroid that contains a clone set with at least three elements, then M is isomorphic to one of* $U_{2,4}$, $U_{2,5}$, $U_{3,5}$, *and* $U_{3,6}$.

Corollary 1.5. *If M is a* 3-*connected* GF(5)-*representable matroid that contains a clone set with at least four elements, then M is isomorphic to one of* $U_{2,4}$, $U_{2,5}$, $U_{2,6}$, $U_{3,5}$, $U_{3,6}$, *and* $U_{4,6}$.

2. The proofs

We define several matroids, give preliminary results, and prove Theorem 1.2 in this section. The matroids P_6, Q_6, and Φ_3^+ pictured have largest clone sets of size three, two, and two, respectively. These matroids are not representable over fields with fewer than five, four, and four elements, respectively. Hence none of these matroids is a counterexample to Theorem 1.2 (Fig. 1).

The rank-k *free spikes* Φ_k and *free swirls* Ψ_k, for $k \geqslant 3$, are matroids on ground set $\{a_1, b_1, a_2, b_2, \dots, a_k, b_k\}$. The nonspanning circuits of Φ_k are the sets $\{a_i, b_i, a_j, b_j\}$ for distinct i and j in $\{1, 2, \dots, k\}$. The nonspanning circuits of Ψ_k are the sets $\{a_i, b_i, e_{i+1}, e_{i+2}, \dots, e_{j-1}, , a_j, b_j\}$ where $e_\ell \in \{a_\ell, b_\ell\}$ for distinct i and j in $\{1, 2, \dots, k\}$ and $\{i, i+1, \dots, j-1, j\} \neq \{1, 2, \dots, k\}$ (subscripts modulo k). Note that $\Phi_3 \cong \Psi_3 \cong U_{3,6}$. If $k \geqslant 4$, then the sets $\{a_i, b_i\}$ are the clone sets of Φ_k and Ψ_k. These spikes and swirls are not counterexamples to Theorem 1.2 as they are not ternary.

The simplification and cosimplification of M are denoted by $\mathrm{si}(M)$ and $\mathrm{co}(M)$. Let N be a 3-connected minor of a 3-connected matroid M with $|E(N)| \geqslant 4$. Consider the following four conditions on $e \in E(M)$.

(1) If $\mathrm{co}(M \backslash e)$ is 3-connected with an N-minor, then e is not fixed in M.
(2) If $\mathrm{si}(M/e)$ is 3-connected with an N-minor, then e is not cofixed in M.
(3) If $\mathrm{co}(M \backslash e)$ is 3-connected, then e is not fixed in M.
(4) If $\mathrm{si}(M/e)$ is 3-connected, then e is not cofixed in M.

If M satisfies conditions (1) and (2) for all $e \in E(M)$, then M is a *totally free expansion* of N. If M satisfies conditions (3) and (4) for all $e \in E(M)$, then M is *totally free*. We next give some preliminary results beginning with a powerful extension of Seymour's Splitter Theorem [8].

Theorem 2.1 (*Geelen et al. [4, Theorem 9.1]*). *Let N be a* 3-*connected minor of a* 3-*connected matroid M with* $|E(N)| \geqslant 4$. *If M is not a wheel or a whirl and M is not a totally free expansion of N, then there is an element e of* $E(M)$ *such that either* $M \backslash e$ *is* 3-*connected with an N-minor and e is fixed in M, or* M/e *is* 3-*connected with an N-minor and e is cofixed in M*.

Corollary 2.2 (*Geelen et al. [4, Corollary 8.6]*). *A matroid M is totally free if and only if it is a totally free expansion of* $U_{2,4}$.

Lemma 2.3 (*Geelen et al. [4, Proposition 4.3]*). *If e and f are clones in a matroid M, then e and f are clones in any minor of M that contains them.*

Theorem 2.4 (*Implicit in Geelen et al. [3]*). *A* GF(3)-*representable matroid M is totally free if and only if M is isomorphic to* $U_{2,4}$.

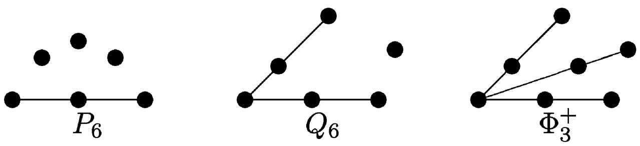

Fig. 1. Rank-3 geometries.

Theorem 2.5 (*Geelen et al.* [4, Theorem 2.5]). *A* GF(4)-*representable matroid* M *is totally free if and only if* M *is isomorphic to one of* $U_{2,4}$, $U_{2,5}$, $U_{3,5}$, *or* Φ_r *for some* $r \geqslant 3$.

Theorem 2.6 (*Geelen et al.* [4, Theorem 2.7]). *A* GF(5)-*representable matroid* M *is totally free if and only if* M *is isomorphic to one of* $U_{2,4}$, $U_{2,5}$, $U_{2,6}$, $U_{3,5}$, $U_{4,6}$, P_6 *or* Ψ_r *for some* $r \geqslant 3$.

Proof of Theorem 1.2. Let M be a GF(q)-representable matroid that is a minimal counterexample to the theorem statement. If M is binary, then the result is vacuously true by Theorem 1.1, so $q \neq 2$. The matroids listed in Theorems 2.4, 2.5, and 2.6 are not counterexamples to Theorem 1.2, so M is not totally free. Moreover, by Corollary 2.2, M is not a totally free expansion of $U_{2,4}$.

The matroid M contains a clone set and so is neither a wheel nor a whirl. It follows from applying Theorem 2.1 with $N \cong U_{2,4}$ and Corollary 2.2 that M is either an extension or a coextension of a 3-connected nonbinary matroid T by an element e that is either fixed or cofixed, respectively, in M. Evidently $e \notin X$ as clone sets do not contain fixed elements. It follows from Lemma 2.3 and T being 3-connected that X is a clone set of T. The choice of M implies that T is uniform.

We may assume that $M \backslash e = T$ by duality. If $r(T) = 2$, then $r(M) = 2$, so M is uniform; a contradiction. Hence $r(T) \geqslant 3$. Let $S(\ell)$ be the set of uniform matroids with rank exceeding two that are 3-connected, GF(ℓ)-representable, and nonbinary. Then $T \in S(q)$. The set $S(3) = \emptyset$, so $q \neq 3$. Suppose $q = 4$. The set $S(4) = \{U_{3,5}, U_{3,6}\}$. The only 3-connected nonuniform GF(4)-representable extensions of these matroids are Q_6 and Φ_3^+; a contradiction. Thus, $q \neq 4$. Hence $q = 5$. The set $S(5) = \{U_{3,5}, U_{3,6}, U_{4,6}\}$. The only 3-connected nonuniform extensions of the first two matroids in this list are P_6, Q_6, P_6^+, Q_6^+, and Φ_3^+ where the matroids P_6^+ and Q_6^+ are obtained by freely adding an element to P_6 and Q_6, respectively. The matroid P_6^+ is not GF(5)-representable [4, Lemma 11.1], while the other matroids do not have sufficiently large clone sets. Thus M is an extension of $U_{4,6}$. Hence M^* is a coextension of $U_{2,6}$ that contains a clone set of size at least four. There is a line L of M^* that contains at least three points as this matroid is nonuniform. Either X is contained in L or not. In the former case, M^* is a rank-three matroid that is the union of the four-point line L and three free points. However, this matroid is not GF(5)-representable [2]. In thelatter case, M^* is a rank-three matroid that is the union of a three-point line and four free points. Hence $M^* \cong P_6^+$; again a contradiction. □

A longer proof of Theorem 1.2 that avoids the use of Theorem 2.1 may be given using Bixby's [1] result that if e is an element of a 3-connected matroid M, then either $\mathrm{si}(M/e)$ or $\mathrm{co}(M \backslash e)$ is 3-connected.

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